

Solution for 'Topics in complex analysis'

(03/12/2025)

H 12.1 (The Riemann sphere $\widehat{\mathbb{C}}$)

The goal of this exercise is to introduce calculus at ∞ . We set $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where for the moment ∞ is an abstract element. We say that a sequence $\{z_n\} \subset \widehat{\mathbb{C}}$ converges to $z \in \widehat{\mathbb{C}}$ if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\begin{cases} z_n = \infty & \text{or} & |z_n| \geq \varepsilon^{-1} & \text{if } z = \infty, \\ z_n \in \mathbb{C} & \text{and} & |z_n - z| \leq \varepsilon & \text{if } z \in \mathbb{C}. \end{cases}$$

Set $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$. Show that the stereographic projection $P : \mathbb{S}^2 \rightarrow \widehat{\mathbb{C}}$ defined by

$$P(x_1, x_2, x_3) = \begin{cases} \frac{x_1}{1-x_3} + i \frac{x_2}{1-x_3} & \text{if } x_3 \neq 1, \\ \infty & \text{otherwise} \end{cases}$$

is bijective, and that both P and its inverse P^{-1} are (sequentially) continuous. Conclude that $\widehat{\mathbb{C}}$ is (sequentially) compact.

Remark: Observe that the definition of convergence given above coincides with the one induced by the topology of $\widehat{\mathbb{C}}$ described in the lecture notes.

Solution H 12.1:

We first show injectivity. Note that if $P(x) = P(y)$, then either $P(x) = \infty$, so that $x = y = (0, 0, 1)$, or $x_3, y_3 \neq 1$ and we have

$$\frac{x_1}{1-x_3} = \frac{y_1}{1-y_3} \quad \text{and} \quad \frac{x_2}{1-x_3} = \frac{y_2}{1-y_3}.$$

Squaring both equalities and summing them yields

$$\frac{1-x_3^2}{(1-x_3)^2} = \frac{x_1^2+x_2^2}{(1-x_3)^2} = \frac{y_1^2+y_2^2}{(1-y_3)^2} = \frac{1-y_3^2}{(1-y_3)^2},$$

where we used that $x_1^2 + x_2^2 + x_3^2 = 1$ and $y_1^2 + y_2^2 + y_3^2 = 1$. We can further simplify the display above to deduce that

$$\frac{1+x_3}{1-x_3} = \frac{1+y_3}{1-y_3}.$$

Note that the function $t \mapsto \frac{1+t}{1-t}$ is strictly increasing on $[-1, 1)$, so that the above equality yields $x_3 = y_3$ and we conclude that $x = y$. Hence P is injective.

Next let $z \in \mathbb{C}$. We show that there exists $x \in \mathbb{S}^2$ such that $P(x) = z$ (the case $z = \infty$ is obvious). Since $|P(x)|^2 = (1+x_3)(1-x_3)^{-1}$ we set $x_3 = (|z|^2 - 1)(1 + |z|^2)^{-1}$. Set also

$$\begin{aligned} x_1 &= \operatorname{Re}(z) \left(1 - \frac{|z|^2 - 1}{1 + |z|^2} \right) = \operatorname{Re}(z) \frac{2}{1 + |z|^2}, \\ x_2 &= \operatorname{Im}(z) \left(1 - \frac{|z|^2 - 1}{1 + |z|^2} \right) = \operatorname{Im}(z) \frac{2}{1 + |z|^2}. \end{aligned}$$

Then by a direct calculation $x_1^2 + x_2^2 + x_3^2 = 1$ and $P(x) = z$. In particular, the inverse function is given by

$$P^{-1}(z) = \begin{cases} \frac{1}{1+|z|^2}(2\operatorname{Re}(z), 2\operatorname{Im}(z), |z|^2 - 1) & \text{if } z \in \mathbb{C}, \\ (0, 0, 1) & \text{if } z = \infty. \end{cases}$$

We next show that P and P^{-1} are sequentially continuous. Let $x_n \rightarrow x$ in \mathbb{S}^2 . If $x \neq (0, 0, 1)$ then clearly $P(x_n) \rightarrow P(x)$. If $x = (0, 0, 1)$, we may assume without loss of generality that $x_n \neq x$ for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} |P(x_n)| = +\infty$, which shows that $P(x_n) \rightarrow \infty$. Thus P is continuous. Now assume that $z_n \rightarrow z$ in $\widehat{\mathbb{C}}$. If $z \in \mathbb{C}$ then again it is clear that $P^{-1}(z_n) \rightarrow P^{-1}(z)$. If $z = \infty$ we can assume as before that $z_n \neq \infty$ for all $n \in \mathbb{N}$. Then by definition $\lim_{n \rightarrow \infty} |z_n| = +\infty$, so that $\lim_{n \rightarrow \infty} P^{-1}(z_n) = (0, 0, 1)$ since $|\operatorname{Re}(z)|, |\operatorname{Im}(z)| \leq |z|$.

In order to show sequential compactness, it suffices to note that $\widehat{\mathbb{C}}$ is the image of the sequentially compact set \mathbb{S}^2 under the continuous function P . □

H 12.2 (The open mapping theorem for $\widehat{\mathbb{C}}$)

Let $\widehat{D} \subset \widehat{\mathbb{C}}$ be a domain and let $f : \widehat{D} \rightarrow \widehat{\mathbb{C}}$ be holomorphic and non-constant. Show that $f(\widehat{D})$ is again a domain.

Solution H 12.2:

Since f is continuous it follows that $f(\widehat{D})$ is again a path-connected set. Hence we have to show that $f(\widehat{D})$ is an open set. Let $w \in f(\widehat{D})$. If $w \in \mathbb{C}$ and there exists $z \in \widehat{D} \setminus \{\infty\}$ such that $f(z) = w$ then by continuity there exists $r > 0$ such that $B_r(z) \subset \widehat{D}$ and $f(B_r(z)) \subset \mathbb{C}$. By the identity theorem on $\widehat{\mathbb{C}}$ it follows that f is not constant on $B_r(z)$. Hence by the standard open mapping theorem there exists $\varepsilon > 0$ such that $B_\varepsilon(w) \subset f(B_r(z)) \subset f(\widehat{D})$.

When $w = \infty$ and there exists $z \in \widehat{D} \setminus \{\infty\}$ such that $f(z) = w$ we repeat the argument with the non-constant, holomorphic function $1/f : B_r(z) \rightarrow \mathbb{C}$, which yields that there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subset 1/f(B_r(z))$ (recall that $1/\infty = 0$). Rearranging terms, this yields that $\widehat{\mathbb{C}} \setminus \overline{B_{1/\varepsilon}(0)} \subset f(B_r(z)) \subset f(\widehat{D})$. The left hand side set is an open neighborhood of ∞ .

If $w \in \mathbb{C}$ and $f(\infty) = w$ we consider the holomorphic, non-constant function $g : B_r(0) \rightarrow \mathbb{C}$ given by $g(z) = f(\frac{1}{z})$, which is well-defined for r small enough since \widehat{D} is open. Again we deduce that there exists $\varepsilon > 0$ such that $B_\varepsilon(w) \subset f(\widehat{D})$.

In the remaining case $w = \infty = f(\infty)$ we consider the function $g : B_r(0) \rightarrow \mathbb{C}$ given by $g(z) = 1/f(\frac{1}{z})$, for $r > 0$ sufficiently small. Then there is $\varepsilon > 0$ such that $B_\varepsilon(0) \subset g(B_r(0))$, which yields that $\widehat{\mathbb{C}} \setminus \overline{B_{1/\varepsilon}(0)} \subset f(\widehat{D})$.

In all four cases we concluded that there exists a neighborhood N of w in $\widehat{\mathbb{C}}$ such that $N \subset f(\widehat{D})$. This concludes the proof. □

H 12.3 (On the extension of entire functions)

In this exercise we prove that polynomials are the only entire functions that have a holomorphic extension to the Riemann sphere.

a) Let $P : \mathbb{C} \rightarrow \mathbb{C}$ be a non-constant polynomial. Show that setting $P(\infty) := \infty$ defines a holomorphic extension $P : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$.

b) Show that if $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is holomorphic and satisfies $f(\mathbb{C}) \subset \mathbb{C}$ then f is a polynomial.

Hint: Consider $z \mapsto f(1/z)$ and its singularity at 0.

Solution H 12.3:

a) Let $P(z) = a_0 + a_1z + \dots + a_nz^n$ with $a_n \neq 0$ and $n \geq 1$. Since we set $P(\infty) = \infty$, in order

to prove that P is holomorphic at ∞ we have to consider the function $g(z) = 1/P\left(\frac{1}{z}\right)$ near 0. Observe that $g(0) = 0$, so the difference quotient at 0 reads

$$\frac{g(z) - g(0)}{z} = \frac{1}{P\left(\frac{1}{z}\right)z} = \frac{1}{a_0z + a_1 + \dots + a_nz^{1-n}} \xrightarrow{z \rightarrow 0} \begin{cases} 0 & \text{if } n > 1, \\ \frac{1}{a_1} & \text{if } n = 1. \end{cases}$$

Hence P is complex-differentiable at ∞ , and therefore holomorphic on $\widehat{\mathbb{C}}$.

b) Assume by contradiction that f is not a polynomial. By the Taylor series expansion of f at 0, for any $z \in \mathbb{C} \setminus \{0\}$ we have

$$f\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} a_k z^{-k}$$

with infinitely many $a_k \neq 0$. In particular, $z = 0$ is an essential singularity of $z \mapsto f\left(\frac{1}{z}\right)$. By Picard's great theorem there exist two distinct $w_1, w_2 \in \mathbb{C}$ and two sequences $z_n^1, z_n^2 \in \mathbb{C} \setminus \{0\}$ such that $z_n^1, z_n^2 \rightarrow 0$ as $n \rightarrow +\infty$ with $f(1/z_n^1) = w_1$ and $f(1/z_n^2) = w_2$ for all $n \in \mathbb{N}$. But the continuity of f at ∞ implies that

$$f(\infty) = \lim_{n \rightarrow +\infty} f\left(\frac{1}{z_n^1}\right) = w_1 \neq w_2 = \lim_{n \rightarrow +\infty} f\left(\frac{1}{z_n^2}\right) = f(\infty).$$

This yields a contradiction. □

H 12.4 (Holomorphic functions on $\widehat{\mathbb{C}}$ are rational)

Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be holomorphic. Show that f is a rational function, i.e. that there exist polynomials $P, Q : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$f(z) = \frac{P(z)}{Q(z)} \quad \forall z \in \mathbb{C} \setminus \{f = \infty\}.$$

Hint: You may need the following generalization of Liouville's theorem: if $g : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and there exist $R > 0$ and $n \in \mathbb{N}$ such that $|g(z)| \leq R|z|^n$ for all $z \in \mathbb{C}$ with $|z| \geq R$, then g is a polynomial of degree at most n . We showed an even stronger version of this result in the proof of Hadamard's theorem.

Solution H 12.4:

If f is constant there is nothing to prove. Moreover, by Exercise H 12.3 we may assume without loss of generality that $f(z) = \infty$ for some $z \in \mathbb{C}$. Since $\widehat{\mathbb{C}}$ is compact, the identity theorem on $\widehat{\mathbb{C}}$ implies that $f^{-1}(\infty) \cap \mathbb{C} = \{a_1, \dots, a_n\}$ for some finite $n \in \mathbb{N}$. At each a_i the non-constant holomorphic function $1/f$ has a zero of some order $k_i \in \mathbb{N}$. This implies that the function f has a pole of order k_i at a_i . Define the polynomial

$$Q(z) = \prod_{i=1}^n (z - a_i)^{k_i}.$$

Then $f \cdot Q : \mathbb{C} \setminus \{a_1, \dots, a_n\} \rightarrow \mathbb{C}$ is holomorphic and each singularity is removable. Hence it can be extended to an entire function. If $f(\infty) \in \mathbb{C}$, then there exists $R > 0$ such that $|f(z)| \leq R$ for all $|z| \geq R$. The generalization of Liouville's theorem in the hint then implies that $f \cdot Q$ is a polynomial.

Otherwise $f(\infty) = \infty$, and setting $f \cdot Q(\infty) := \infty$ defines a holomorphic extension to $\widehat{\mathbb{C}}$ of the entire function $f \cdot Q : \mathbb{C} \rightarrow \mathbb{C}$. Indeed, this is just a reformulation of the product rule, since by Exercise H 12.3 a) the (non-constant) polynomial Q is holomorphic at ∞ with value ∞ . From H 12.3 b) we deduce again that $f \cdot Q$ is a polynomial.

Remark: We distinguished the two cases $f(\infty) = \infty$ and $f(\infty) \in \mathbb{C}$ because of the possibility that $f(\infty) = 0$, in which case simply setting $f \cdot Q(\infty) = \infty$ may fail to give a holomorphic extension. The underlying issue is that in general the product of two functions with values in $\widehat{\mathbb{C}}$ is not well-defined. \square